Now,

$$\frac{dv}{dr} = \pi h\left[h + (2r - h)\frac{dh}{dr}\right], \qquad \text{and} \qquad \frac{dh}{dr} = 1 - \csc\alpha.$$

Therefore,

$$\frac{dv}{dr} = \pi h[a\csc\alpha - r(\csc\alpha - 1)(\csc\alpha + 2)].$$

Obviously, $h = 0$ cannot give a maximum overflow. Consequently,

$$r = \frac{a\csc\alpha}{(\csc\alpha - 1)(\csc\alpha + 2)} = \frac{a\sin\alpha}{(1 - \sin\alpha)(1 + 2\sin\alpha)} = a\sin\alpha/(\sin\alpha + \cos 2\alpha).$$

Furthermore,

$$\frac{d^2v}{dr^2} = -\pi h(\csc\alpha - 1)(\csc\alpha + 2).$$

Since $h \neq 0$ and $\csc\alpha > 1$, d^2v/dr^2 is negative so that the condition for a maximum is satisfied.

Also solved by A. H. Wilson, S. E. Rasor, Elmer Schuyler, George Raynor, C. N. Schmall, H. L. Agard, C. N. Mills, Lewis Clark, L. M. Pickett, C. A. Bergstresser, C. Hornung, L. G. Weld, J. A. Bullard, G. W. Hartwell, J. V. Balch, J. A. Whitted, Elijah Swift, J. C. Rayworth, and J. A. Caparo.

396. Proposed by ELBERT H. CLARKE, Purdue University.

The length of the curve $y = x^n$ from the origin to the point (1, 1) is given by the formula

$$= \int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this length as n becomes infinite is 2. Give a strict analytic proof that

$$\lim_{n\to\infty}\int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx = 2.$$

Solution by Elijah Swift, University of Vermont.

We can easily show that

(1) $$\lim_{n\to\infty} n^2x^{2n-2} = 0, \qquad 0 \leqq x < 1.$$

Next, we choose a positive number, ϵ, as small as we like. We can then find a value N of n such that for it and all larger values

(2) $$n^2(1 - \epsilon)^{2n-2} < \eta,$$

where η is a positive number as small as we like. This is possible on account of (1).

Let x_n be a value of x such that $n^2{x_n}^{2n-2} = 2$.

Now write the given integral, I, as a sum of three integrals.

$$I = \int_0^{1-\epsilon}\sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{1-\epsilon}^{x_n}\sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{x_n}^1\sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Considering values of $n > N$, we have

(3) $$\left|\int_0^{1-\epsilon}\sqrt{1 + n^2x^{2n-2}}\cdot dx - 1\right| = \left|\int_0^{1-\epsilon}\{\sqrt{1 + n^2x^{2n-2}} - 1\}\,dx - \epsilon\right|$$
$$\leqq \left|\int_0^{1-\epsilon}\{\sqrt{1 + \eta} - 1\}\,dx\right| + \epsilon < \frac{\eta}{2} + \epsilon. \quad \text{(See (2).)}$$

This holds if $n \geqq N$.

Now

(4) $$\int_{1-\epsilon}^{x_n}\sqrt{1 + n^2x^{2n-2}}\cdot dx < \int_{1-\epsilon}^1\sqrt{1 + 2}\cdot dx = \sqrt{3}\cdot\epsilon.$$

Finally, since for values of x greater than x_n, $n^2x^{2n-2} > 2$, we may develop $\sqrt{1 + n^2x^{2n-2}}$ into a series in descending powers of n^2x^{2n-2}, and the series will be uniformly and absolutely convergent for values of x between, and including, x_n and 1, and all values of $n \geqq N$, we have

$$\sqrt{1 + n^2x^{2n-2}} = nx^{n-1} + \frac{1}{2} \cdot \frac{1}{nx^{n-1}} - \frac{1}{2 \cdot 4} \cdot \frac{1}{n^3x^{3n-3}} + \cdots.$$

Hence,

(5) $$\int_{x_n}^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 - x_n{}^n + \left[\frac{1}{2} \cdot \frac{1}{n(-n+2)} x^{-n+2} - + \cdots\right]_{x_n}^1.$$

Since,

(6) $$nx_n{}^{n-1} = \sqrt{2},$$

(7) $$\lim_{n \to \infty} x_n{}^n = 0.$$

We may now obtain the limit of the right hand side of (5) when $n \to \infty$ by taking the limit of each term. This gives, in consequence of (6) and (7), 1 as the limit. In other words, we may choose n so large that

(8) $$\int_{x_n}^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 + \zeta,$$

where ζ is a constant as small numerically as we please for all values of $n > N$.

Then

$$\left|\int_0^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx - 2\right| \leqq \left|\int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1\right| + \left|\int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}} \cdot dx\right|$$
$$+ \left|\int_{x_n}^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1\right| < \frac{\eta}{2} + \epsilon + \sqrt{3} \cdot \epsilon + |\zeta|.$$

Since this may be made as small as we please by a suitable choice in order of ϵ, η, and ζ, we have proved the statement.

Also solved by TOBIAS DANTZIG.

MECHANICS.

312. Proposed by C. N. SCHMALL, New York City.

A ball of elasticity e is projected upward from a point on an inclined plane, so that after its first contact with the plane it rebounds to its starting point. If ϕ be the inclination of the plane to the horizontal, and ψ the angle made by the line of projection with the inclined plane, show that

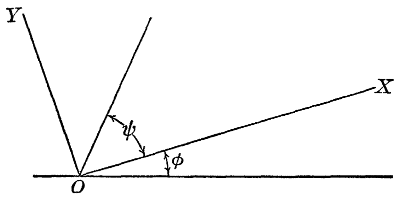
$$\cot \phi \cot \psi = e + 1.$$

I. SOLUTION BY HORACE OLSON, Chicago, Illinois.

In order that the answer given may be true the ball must be thrown in a vertical plane perpendicular to the inclined plane. Taking the axis of coordinates as indicated in the figure, the equations of the trajectory are:

$$x = vt \cos \psi - \frac{gt^2 \sin \phi}{2},$$

$$y = vt \sin \psi - \frac{gt^2 \cos \phi}{2},$$



t being the time from the instant of projection. Then,

$$\frac{dx}{dt} = v \cos \psi - gt \sin \phi \quad \text{and} \quad \frac{dy}{dt} = v \sin \psi - gt \cos \phi.$$